MOREILLON Philippe N°F2940 Session 8

1/

[T.K] 8.5.1.3 Determine c so that $c \cdot e^{-(x^2-xy+y^2)}$ becomes the p.d.f of a bivariate normal distribution and determine its mean vector and covariance matrix.

Remmeder : if (X,Y) is a bivariate normal vector, then $f_{(X,Y)}(x,y) = \frac{1}{2\pi\sqrt{\det C}} e^{-\frac{1}{2}\left(\binom{x}{y}-E[\binom{X}{Y}]\right)^T C^{-1}\left(\binom{x}{y}-E[\binom{X}{Y}]\right)}$ $\forall x,y \in \mathbb{R}$.

1/ Determine C^{-1}

$$(x^2-xy+y^2) = \binom{x}{y}^T\begin{pmatrix}1 & -1/2\\ -1/2 & 1\end{pmatrix}\binom{x}{y} \Rightarrow -(x^2-xy+y^2) = -\frac{1}{2}\binom{x}{y}^T\begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}\binom{x}{y} \quad (*)$$

$$\Rightarrow C^{-1} = \begin{pmatrix}2 & -1\\ -1 & 2\end{pmatrix}$$

2/ Deduce c and the other quantities :

$$\det(C) = \frac{1}{\det(C^{-1})} = \frac{1}{3} \Rightarrow c = \frac{1}{2\pi\sqrt{\frac{1}{3}}} = \frac{\sqrt{3}}{2\pi} \quad //$$

$E[\binom{X}{Y}] = \binom{0}{0}$, (there was no constant in the initial expression : $x^2-xy+y^2 \Rightarrow$ the term $E[\binom{X}{Y}]$ is 0 in the factorisation $(*)$)

$$C = \mathrm{cov}(X,Y) = \frac{1}{\det(C^{-1})}\begin{pmatrix}2 & 1\\ 1 & 2\end{pmatrix} = \begin{pmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{pmatrix}$$

Cramer's rule : $A = \begin{pmatrix}a & b\\ c & d\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det A}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

$$\Rightarrow \binom{X}{Y} \in \mathcal{N}\left(\binom{0}{0}, \begin{pmatrix}2/3 & 1/3\\ 1/3 & 2/3\end{pmatrix}\right)$$

[T.K] 8.5.1.10

Let $\binom{Y_1}{Y_2} \in \mathcal{N}\left(\binom{0}{0}, \begin{pmatrix}1+\rho & 0\\ 0 & 1-\rho\end{pmatrix}\right)$ and set $\binom{X_1}{X_2} = Q\binom{Y_1}{Y_2}$, where $Q = \begin{pmatrix}\cos\pi/4 & -\sin\pi/4\\ \sin\pi/4 & \cos\pi/4\end{pmatrix}$

To show : $\binom{X_1}{X_2} = Q\binom{Y_1}{Y_2} \in \mathcal{N}\left(\binom{0}{0}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix}\right)$

We will see that the distribution is preserved via rotations and how it affects its parameters. Let $A \in \mathcal{B}(\mathbb{R}\times\mathbb{R})$,

$$P\left(\binom{X_1}{X_2}\in A\right) = P\left(Q\binom{Y_1}{Y_2}\in A\right) = P\left(\binom{Y_1}{Y_2}\in Q^{-1}A\right) = \iint_{Q^{-1}A} f_{(Y_1,Y_2)}(y_1,y_2)\,dy_1\,dy_2$$

$$= \iint_{Q^{-1}A}\frac{1}{2\pi\sqrt{\det C}} e^{-\frac{1}{2}\left(\binom{y_1}{y_2}-E[\binom{Y_1}{Y_2}]\right)^T C^{-1}\left(\binom{y_1}{y_2}-E[\binom{Y_1}{Y_2}]\right)}\,dy_1\,dy_2$$

$$= \iint_{A}\frac{1}{2\pi\sqrt{\det C}} e^{-1/2\left(\binom{x_1}{x_2}-E[\binom{X_1}{X_2}]\right)^T(Q^{-1})^T C^{-1}Q^{-1}\left(\binom{x_1}{x_2}-E[\binom{X_1}{X_2}]\right)}\,|\det Q|\,dx_1\,dx_2$$

$\binom{X_1}{X_2} = Q\binom{Y_1}{Y_2}$

$E[Q\binom{Y_1}{Y_2}] = Q\,E[\binom{Y_1}{Y_2}]$

$$= \iint_{A}\frac{1}{2\pi\sqrt{\det(QCQ^T)}} e^{-1/2\left(\binom{x_1}{x_2}-E[\binom{X_1}{X_2}]\right)^T(QCQ^T)^{-1}\left(\binom{x_1}{x_2}-E[\binom{X_1}{X_2}]\right)}\,dx_1\,dx_2$$

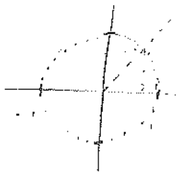
$\det Q = 1$

$(Q^{-1})^T = (Q^T)^{-1}$

$\Rightarrow$ if $\binom{Y_1}{Y_2} \in \mathcal{N}\left(E[\binom{Y_1}{Y_2}], C\right)$, then $\binom{X_1}{X_2} \in \mathcal{N}\left(Q\,E[\binom{Y_1}{Y_2}], QCQ^T\right)$.

We just have to compute QCQ^T :

First $Q = \begin{pmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$



$\cos \pi/4 = \sin \pi/4$
$\cos \pi/4^2 + \sin \pi/4^2 = 1 \Rightarrow \cos \pi/4 = \sin \pi/4 = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$

$$\Rightarrow QCQ^T = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \begin{pmatrix} \sqrt{2}/2 & +\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\rho & 1+\rho \\ \rho-1 & 1-\rho \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 & 2\rho \\ 2\rho & 2 \end{pmatrix} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \in \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \right)$$

[T.K] 8.5.1.13 Let $X \in \mathcal{N}(0,1)$ and $Y \in \mathcal{N}(0,1)$ be independent.

To show : $\mathbb{E}[X \mid X>Y] = \frac{1}{\sqrt{\pi}}$

$\mathbb{E}[X+Y \mid X>Y] = 0$

$$\mathbb{E}[X \mid X>Y] = \frac{\mathbb{E}[X \mathbb{1}_{\{X>Y\}}]}{\mathbb{P}(X>Y)} \underset{\mathbb{P}(X>Y) = \frac{1}{2}}{=} 2\, \mathbb{E}[X \mathbb{1}_{\{X>Y\}}]$$

$$\mathbb{E}[X \mathbb{1}_{\{X>Y\}}] = \int_{-\infty}^{+\infty} \int_{y}^{+\infty} x\, f_X(x) f_Y(y)\, dx\, dy = \int_{-\infty}^{+\infty} \int_{y}^{+\infty} x \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dx\, dy$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-y^2/2} \underbrace{\int_{y}^{+\infty} x\, e^{-x^2/2}\, dx}_{-e^{-x^2/2} \Big|_y^{+\infty} = e^{-y^2/2}}\, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-y^2/2} \cdot e^{-y^2/2}\, dy = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-y^2}\, dy$$

Now $\frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-y^2} dy \underset{\text{law of } \mathcal{N}(0,2)}{=} 1 \Rightarrow \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-y^2} dy = \frac{1}{2\pi} \cdot \frac{\sqrt{2\pi}}{\sqrt{2}} = \frac{1}{2\sqrt{\pi}}$

Hence $\mathbb{E}[X \mid X>Y] = \frac{1}{\sqrt{\pi}}$ //

* $\mathbb{E}[X+Y \mid X>Y] = 0$.

Heuristics : $\mathbb{E}[X+Y] = \mathbb{E}[X+Y \mid X>Y]\, \mathbb{P}(X>Y) + \mathbb{E}[X+Y \mid X<Y]\, \mathbb{P}(X<Y)$

$$= \frac{1}{2}\left(\mathbb{E}[X+Y \mid X>Y] + \mathbb{E}[X+Y \mid X<Y]\right)$$

Now $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 0$.

Moreover we would like to say that $\mathbb{E}[X+Y \mid X>Y] = \mathbb{E}[X+Y \mid X<Y]$. This looks obvious since X and Y are two independent representants of the same laws $\Rightarrow \{X>Y\}$ and $\{Y>X\}$ should impact $X+Y$ equally.

Let's prove this : $\mathbb{E}[X+Y \mid X>Y] = 2\, \mathbb{E}[(X+Y)\mathbb{1}_{\{X>Y\}}] = 2 \int_{-\infty}^{+\infty}\int_{y}^{+\infty} (x+y) f_X(x) f_Y(y)\, dx\, dy$

$$\underset{f_X = f_Y}{=} 2 \int_{-\infty}^{+\infty}\int_{y}^{+\infty} (x+y) f_Y(x) f_X(y)\, dx\, dy = 2\, \mathbb{E}[(X+Y)\mathbb{1}_{\{Y>X\}}] = \mathbb{E}[X+Y \mid Y>X].$$

Hence, $\mathbb{E}[X+Y \mid X>Y] = \mathbb{E}[X+Y \mid X<Y] = 0$.

[T.K] 8.5.1.15 let $X_1, X_2, X_3 \in \mathcal{N}(1,1)$ be independent.

Set $U = X_1 + X_2 + X_3$ and $V = X_1 + 2X_2 + 3X_3$.

Determine $V \mid U=3$.

By the course, $\begin{pmatrix} U \\ V \end{pmatrix}$ has a multivariate normal distribution iff $\forall a_1, a_2 \in \mathbb{R}$, $a_1 U + a_2 V$ is normal.

Here $a_1 U + a_2 V = (a_1 + a_2) X_1 + (a_1 + 2a_2) X_2 + (a_1 + 3a_2) X_3$ and we know that the linear combination of independent normal random variables is normal (just see that via the characteristic function).

Now $\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \Rightarrow \mathbb{E}\left[\begin{pmatrix} U \\ V \end{pmatrix}\right] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \mathbb{E}\left[\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right] = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$

$$\operatorname{Cov}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix} \underbrace{\operatorname{Cov}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}}_{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} U \\ V \end{pmatrix} \in \mathcal{N}\left(\begin{pmatrix} 3 \\ 6 \end{pmatrix}, \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\right)$

and $U \in \mathcal{N}(3,3)$

$$\Rightarrow f_{V \mid U=3}(v) = \frac{f_{(U,V)}(3,v)}{f_U(3)} = \frac{\frac{1}{2\pi\sqrt{6}} \exp\left(-\frac{1}{2}\left(\begin{pmatrix} 3 \\ v \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix}\right)^T \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\left(\begin{pmatrix} 3 \\ v \end{pmatrix} - \begin{pmatrix} 3 \\ 6 \end{pmatrix}\right)\right)}{\frac{1}{\sqrt{2\pi}\sqrt{3}} \exp\left(-\frac{(3-3)^2}{2\cdot 3}\right)}$$

$$\det\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix} = 6$$

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}$$

$$= \frac{\sqrt{2\pi}\sqrt{3}}{2\pi\sqrt{6}} \exp\left(-\frac{1}{4}(v-6)^2\right) = \frac{1}{\sqrt{2\pi}\cdot\sqrt{2}} \exp\left(-\frac{1}{2}\frac{(v-6)^2}{2}\right)$$

$\Rightarrow V \mid U=3 \in \mathcal{N}(6,2)$

[T-K] 8.5.1.16 Let $(X_1, X_2, X_3)'$ be a multivariate Gaussian vector with $\mathbb{E}\left[\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and

$$cov\left(\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right) = \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Find the distribution of $X_1 + X_3$ given that : a.) $X_2 = 0$

b.) $X_2 = 2$

(a) Let $U = X_1 + X_3$. We want to determine $f_{U|X_2=0}(u) = \dfrac{f_{(U,X_2)}(u,0)}{f_{X_2}(0)}$

First remark that $X_2 \in \mathcal{N}(0,2)$. Indeed $\forall \alpha, \beta, \gamma \in \mathbb{R}$, $\alpha X_1 + \beta X_2 + \gamma X_3$ is normally distributed and $\mathbb{E}[X_2] = 0$ and $Var(X_2) = 2$. $\Rightarrow f_{X_2}(0) = \dfrac{1}{\sqrt{2\pi}\sqrt{2}} e^{-0^2/2\cdot 2} = \dfrac{1}{\sqrt{2\pi}\sqrt{2}}$.

Now $\begin{pmatrix} U \\ X_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \Rightarrow \mathbb{E}\left[\begin{pmatrix} U \\ X_2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \mathbb{E}\left[\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$cov\left[\begin{pmatrix} U \\ X_2 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} cov\left(\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}\right) \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ -2 & 2 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}$$

As before, $\begin{pmatrix} U \\ X_2 \end{pmatrix}$ is normally distributed

$$\Rightarrow f_{(U,X_2)}(u,0) = \frac{1}{2\pi\sqrt{8}} \exp\left(\frac{1}{2}\begin{pmatrix} u \\ 0 \end{pmatrix}^T \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ 0 \end{pmatrix}\right) = \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\frac{1}{4}u^2\right)$$

$$\det\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix} = 8$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 2 \end{pmatrix}^{-1} = \frac{1}{8}\begin{pmatrix} 2 & 2 \\ 2 & 6 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$$

$$\Rightarrow f_{U|X_2=0}(u) = \frac{\sqrt{2\pi}\sqrt{2}}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}u^2\right) = \frac{1}{\sqrt{2\pi}\cdot 2} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}u^2\right) \Rightarrow U|X_2=0 \in \underline{\mathcal{N}(0,4)}$$

(b) $f_{X_2}(2) = \dfrac{1}{\sqrt{2\pi}\sqrt{2}} e^{-\frac{1}{2\cdot 2}(2-0)^2} = \dfrac{1}{\sqrt{2\pi}\sqrt{2}} e^{-1}$

$$f_{(U,X_2)}(u,2) = \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\begin{pmatrix} u \\ 2 \end{pmatrix}^T \frac{1}{4}\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} u \\ 2 \end{pmatrix}\right) = \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}\begin{pmatrix} u \\ 2 \end{pmatrix}^T \begin{pmatrix} u+2 \\ u+6 \end{pmatrix}\right)$$

$$= \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}\, u^2 + 2u + 2u + 12\right) = \frac{1}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}(u^2 + 4u + 12)\right)$$

$$\Rightarrow f_{U|X_2=2}(u) = \frac{\sqrt{2\pi}\sqrt{2}}{2\pi\sqrt{8}} \exp\left(-\frac{1}{2}\cdot\frac{1}{4}(u^2+4u+12) + 1\right) = \frac{1}{\sqrt{2\pi}\cdot 2} \exp\Big(-\frac{1}{2}\cdot\frac{1}{4}\underbrace{(u^2+4u+12-8)}_{(u+2)^2}\Big) \Rightarrow U|X_2=2 \in \underline{\mathcal{N}(-2,4)}$$